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From SYM perturbation theory to closed strings in Matrix theory

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Abstract: For the purpose of better understanding the AdS/CFT correspondence it is useful to have a description of the theory for all values of the 't Hooft coupling, and for all N. We discuss such a description in the framework of Matrix theory for SYM on D4-branes, which is given in terms of quantum mechanics on the moduli space of solutions of the Nahm equations. This description reduces to both SYM perturbation theory and to closed string perturbation theory, each in its appropriate regime of validity, suggesting a way of directly relating the variables in the two descriptions. For example, it shows explicitly how holes in the world-sheets of the 't Hooft expansion close to give closed surfaces.

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Contents

1. Introduction

In this note we discuss a Matrix model [1] (for a recent review see [2]) for the super-Yang-Mills theory on D4-branes, and show how it transforms the SYM weak coupling Hilbert space and interactions (in DLCQ) into a weakly coupled string theory when N and the 't Hooft coupling become large.

Understanding the relation between gauge theories and strings has been an interesting field of research for some time now ([3], and references therein). A breakthrough in this field is the AdS/CFT correspondence [4], in which the large-N limit of certain gauge theories (or otherwise interacting field theories) was solved in terms of a gravity background. Using this correspondence one can study some properties of theories with gravity and of gauge theories [5, 6] (for a recent review see [7]).

This conjecture has by now been put to the test and verified in numerous ways. Still, one might be interested in a more straightforward way of deriving strings from SYM. Ideally one would like to start perhaps with SYM perturbation theory (i.e. weak coupled SYM where we understanding what are the "fundamental" fields and their interactions) and re-sum the perturbation theory to strong 't Hooft coupling, or begin with Wilson loops, which are again best understood at weak coupling, and again try and extend those to large 't Hooft coupling.

In this paper we will discuss a construction along these lines for the case of SYM in $4+1$ dimensions (with maximal supersymmetry). We will be working at some large but fixed N. The description that we will use is the Matrix description of this theory, which is valid for all values of the 't Hooft coupling and N. This is a quantum mechanical system, which has the interesting property that in the form that we will use it, it can be written in terms of both "closed string" and "SYM" variables, on almost equal footing. Both sets of variables, however, cannot be taken to be dynamical and independent at the same time, as there are constraints which relate the two sets. One can choose only one of these sets of variables to describe the system, giving us either an "open string" (SYM) or an equivalent "closed string" description of the same system. As expected, the "SYM" variables describe the system better at weak 't Hooft coupling where they give the SYM perturbation theory, and the "closed string" variables describe the system at strong 't Hooft coupling and large N where they give weakly coupled closed string perturbation theory on the near-horizon geometry. Again, these are two effective descriptions of the same quantum mechanical system.

Since the AdS/CFT correspondence is a strong/weak coupling duality we do not expect to have a simple map between the Hilbert space of weakly coupled SYM (in DLCQ) and the Hilbert space of the closed strings (in DLCQ). What we will obtain, however, is the best that one can hope for — both SYM and closed strings perturbation theories in DLCQ are shown to be the asymptotic expansions (in different regimes) of the same underlying quantum mechanical system, of which we have a full definition. Hence the duality between them is manifest in this approach.

The organization of the paper is the following. In section 2 we present the DLCQ (Matrix) model for D4-branes. In section 3 we analyze it at weak 't Hooft coupling and briefly explain how SYM perturbation theory is generated in this limit. In section 4 we discuss the model at strong 't Hooft coupling and large N — we review the near-horizon background of a cluster of D4-branes, and then obtain it by a collective coordinate method in the DLCQ description. The method is similar to that used in [8, 9].

Before we proceed one should mention one caveat. SYM with 16 Supercharges in $4+1$ dimensions exists[1] only as the $6\,\mathrm{D}(2,0)$ SCFT compactified on a circle (the size of the circle then sets the SYM coupling) — for a review see [10]. This means that it is not always easy to distinguish gauge theory large 't Hooft coupling effects (i.e. when the loop corrections become strong at energies $1/g_{ym}^2 N$) from effects associate with the strong coupling of the (2,0) field theory (which happens at some energy below g_{ym}^2). This, however, will not play a role in what follows.

This work is reminiscent of ideas presented at [11]. It is also related to ideas in [12], in which closed strings fragment to form a gas of open string bits at the horizon (as in D-branes). The construction presented here is a realization of this idea. We will see the fragmentation quite explicitly, and how open and closed strings transform into each other.

[1]Within field theories. One may also define it using "little string theories".

2. The Matrix model of D4-branes

Matrix theory can be used to describe the DLCQ of field theories on certain solitons.[2] Initially one obtains a Matrix model for the theory on the soliton coupled to gravity [13]. Then, in order to obtain a description of the decoupled theory on the soliton, one traces the space-time decoupling procedure in the Matrix model. The result is a description of the soliton field theory [14]–[18] in terms of a decoupled sector in the Matrix description. As usual in Matrix theory one then needs to identify states with energy that scales like $1/P_{\rm null}$ in the large $P_{\rm null}$ limit (where $P_{\rm null}$ is momentum along the compactified null circle), and only these can be compared with the states of the uncompactified theory. We would like to carry out this procedure for N_4 D4-branes at fixed SYM coupling. By that we mean a cluster of N_4 M5-branes compactified on a fixed circle, whose size is then the $4+1$D SYM coupling. In this section we will present the model, and postpone its analysis in weak and strong 't Hooft coupling to the next two sections.

The Matrix model for M5-branes in flat 11D space is given in terms of the D0-D4 system [13], and that of the compactified M5-branes is given by essentially T-dualizing this system. After T-duality one obtains a system of N_1 wrapped D1-branes intersecting (transversally) N_4 D3-branes with hypermultiplets living at the intersection points. The D1-branes wrap a circle in X^4 direction and the D3-branes span coordinates $X^0 \cdots X^3$, and may be located at different points along the X^4 circle. More precisely, the Matrix model is only the field theory on the D1-branes together with the hypermultiplets. On a generic point of the circle the theory is the standard theory of D1-brane, i.e. it has 16 supercharges. The hypermultiplets live at fixed points on the circle and propagate in time, i.e. they are impurities of codimension 1, and break half of the supersymmetry. A detailed derivation, which we will not present here, is found at [19] (and for the case of the Matrix model for 4D $\mathcal{N}=4$ SYM in [18]). The system appears to be renormalizeable by naive power counting arguments, but beyond that there is no good understanding of its quantum mechanical properties. We will assume that it is a consistent quantum theory.

The bosonic matter content is the following. The bulk (away from the impurities) contains:

1. A U(N_1) gauge field,

2. 3 scalar field X^i in the adjoint of U(N_1). In the language of the D1-branes intersecting the D3-branes, these parameterize the positions of the D1-branes in the spatial directions of the D3-brane. In the Matrix interpretation, these 3 fields parameterize the positions in the 3 non-compact directions of the D4-brane, i.e. the coordinates of the D4-branes other then time and null circle.

[2] Typically, high-dimensional high supersymmetry ones.

3. 5 scalars Y^μ in the adjoint of $\mathrm{U}(N_1)$. These parameterize the coordinates transverse to the D3-branes (or D4-branes in the Matrix interpretation of the model).

These fields are part of a single multiplet from the point of view of the 16 supercharges of the bulk. The full theory, however, has 8 supercharges and the division of the bulk multiplet into the $\mathcal{N} = 8$ multiplets is the following. Recall that in the D0-D4 system the coordinates of the D0-branes parallel to the D4-brane were in an adjoint hypermultiplet. When we T-dualize parallel to the D4-brane, these 4 fields generate the 3 scalar fields X^i and the component of the gauge field A_1, so all of these belong to the same $\mathcal{N} = 8$ multiplet. The remaining fields are the analogue of a vector multiplet as indeed they started their life as a $0+1$ vector multiplet in the D0-D4 system. This structure will manifest itself in the coupling of the various D-terms below.

The impurities are hypermultiplets of the 8 supercharges, which transform as fundamentals of $\mathrm{U}(N_1)$. To specify the impurity data, one specifies some points along the circle which the D1-brane wrap and at each point the number, n_i, of hypermultiplets. These positions correspond to the positions of the D3-branes along the X^4 circle. Each impurity point then has a global symmetry $\mathrm{U}(n_i)$. The total number of D4-branes is Σn_i, and the different positions of the impurities correspond to turning on null Wilson lines in the D4-brane gauge group[3] which breaks it from $\mathrm{U}(\Sigma n_i = N_4)$ to $\Pi\,\mathrm{U}(n_i)$. We will actually restrict ourselves somewhat by turning on null Wilson lines which break the gauge group to a product of U(1)'s, i.e. each hypermultiplet impurity is located at a different point. We will refer to this configuration as the "resolved model". In the large null-momentum limit one expects that these Wilson lines will not play a significant role (although it would be interesting to better understand directly the non-resolved model).

The action that one obtains is the following. The bulk interaction is:

$$\int dt\, d\sigma \Big(\mathcal{D}_t X^i \mathcal{D}_t X^i + \left[Y^\mu, X^i\right]^2 + \frac{1}{g_{ym}^2}\left(F^2 + \mathcal{D}Y^\mu\, \mathcal{D}Y^\mu + [Y^\mu, Y^\nu]^2 + D^{i2}\right) + \\ + \frac{1}{g_{ym}} D^i \left(\mathcal{D}_\sigma X^i + g_{ym}\epsilon_{ijk}\left[X^j, X^k\right]\right)\Big) + \text{ fermion terms}\,, \tag{2.1}$$

where $i = 1,2,3$, $\mu,\nu = 1,\ldots,5$ and $\mathcal{D}$ without a subscript denotes summing over both t and σ. The impurities are located at points σ_k, $k = 1,\ldots,N_4$, and their interaction is

$$\Sigma_k \int dt \Big((\mathcal{D}_t\, Q_k^\alpha)(\mathcal{D}_t\, Q_k^\alpha)^* + Y(\sigma_k)^2\, Q_k\, Q_k^* + D^i(\sigma_k)\, Q_k^\alpha\, \sigma^i_{\alpha\beta}\, Q_k^{\beta *}\Big) + \\ + \text{fermion terms}\,, \tag{2.2}$$

where k is a $\Pi\,\mathrm{U}(n_i)$ index (and we were not careful in lower and upper indices in cases where it is clear how to raise and lower them).

[3] Recall that the brane gauge theory appears as a global symmetry in Matrix theory

The parameters of the $1+1$ theory are related to those of spacetime in the following way. Denoting by Σ the size of the circle of the $1+1$ theory, then

$$\Sigma = \frac{1}{RM_s^2}\,, \qquad g_{ym}^2 = \frac{R^2 M_s^4}{g_s^2} = \frac{R^2 M_s^2}{g_{D4}^4}\,, \tag{2.3}$$

where g_{D4} is the SYM coupling on the D4-brane.

In Matrix theory the coordinates X, Y and Q all measure spacetime distances. This is obvious for the first two, but is also true for Q (because $\sqrt{Q^2}$ is related to a certain instanton size). We will denote the distance coordinates by a subscript -1, and then relation to the fields in (2.1)+(2.2) is

$$X_{-1} = \sqrt{R\Sigma}\,X\,, \qquad Q_{-1} = \sqrt{R}\,Q\,, \qquad Y_{-1} = \frac{\sqrt{R\Sigma}}{g_{ym}}\,Y\,.$$

3. Some comments on the theory at weak 't Hooft coupling

We are interested in discussing the dynamics of the Matrix model in the limit of weak 't Hooft coupling on the D4-brane. Since this coupling is dimensionful we actually require that the effective coupling at the string scale be weak. This translates according to (2.3) to the limit

$$g_{ym} \longrightarrow \infty \tag{3.1}$$

(keeping the energy fixed). In this limit we expect the dynamics to be governed by the moduli space and configurations close to it. In section 3.1 we will identify some configurations which remain in low energies in the open string sector, and in section 3.2 we will discuss how closed+open string perturbation theory comes about. In this section we will be working at a large but fixed N_4.

3.1 The long strings

Given a Matrix model, the states which can be compared to those in the uncompactified theory are those whose energy scales as $1/N_1$ in the large-N_1 limit. In this subsection we will discuss such states in the regime when open string perturbation theory is valid, i.e. at weak 't Hooft coupling. Among these are of course the D4-brane $\mathrm{U}(N_4)$ gauge bosons, and although we will not focus on them specifically in this section, it is clear how to do so.

Let us briefly remind the reader how weakly coupled string theory is derived in the case of the flat 10D IIA string with no additional solitons embedded in it [20, 21]. To find low lying states one begins by identifying appropriate pieces of the moduli space. One then needs to identify small fluctuations around these configurations whose energy has the correct N_1 scaling. Only these can then be compared to the spectrum of the uncompactified theory. In the cases of impurity systems the moduli space is rather complicated and is not understood well enough for our purposes, but one can identify sometimes excitations whose energy scales like $1/N_1$, without a need for a detailed understanding of moduli space.

It is instructive to begin by identifying the states of the single short open string, i.e. the states in the $N_1 = 1$ sector. In this case one can actually identify the entire moduli space. The theory on the D1 is a U(1) gauge theory with 16 supercharges and in addition there are charge 1 hypermultiplets localized at impurities (which break half of the supersymmetries). The Higgs branch is the following. In this case the D-term constraints are

$$\begin{aligned} \partial_\sigma X^i(\sigma) = 0\,, \quad \sigma \neq \sigma_k\,, \\ X^i(\sigma_k+) - X^i(\sigma_k-) = g_{ym}\, Q_k^\alpha\, \sigma^i_{\alpha\beta}\, Q_k^{\beta*}\,, \quad k = 1, \ldots, N_4\,, \end{aligned} \tag{3.2}$$

where by $X(\sigma_k\pm)$ one means the limiting value of X as one approaches the point σ_k from above $(+)$ or from below $(-)$. An excited l-th hypermultiplet means that there are two strings ending on the l-th D4-brane, with opposite charges with respect to the U(1) which lives on that brane, and that the distance between their end points on this brane is given by the 2nd line in (3.2). Going to the Higgs branch, i.e. turning on Q, also makes the Y fields massive. The coordinates of the Higgs branch which we have identified so far give us a space which corresponds to N_4 gauge bosons with total charge 0 in each U(1), which are positioned at different points on R^3. Additional coordinates of the Higgs branch are related to Wilson lines made out of the gauge field A_1. These do not play a role in the weak D4-brane 't Hooft coupling limit, as we will discuss later.

An additional branch is of course the Coulomb branch in which $X, Y \neq 0$ but $Q = 0$. This branch describes short strings in the bulk of the IIA string theory.

There should also be additional "mixed branches", i.e. branches in which the Q's at only part of the impurities are non zero (and zero at the rest). This will have the interpretation of less then N_4 gauge bosons starting and ending on only some of the branes. This however appears as a submanifold in the Higgs branch and it is less clear how to think of it as an additional branch. Fortunately, this problem happens primarily at the $N_1 = 1$ case, and is alleviated in the long string picture, where obtaining the correct spectrum is more critical. We will briefly discuss this later.

Next we identify the configuration of a collection of p long strings, all of which begin and end on the same brane (the generalization to when they begin and end on different branes is straightforward). We will not identify the precise classical zero energy (flat directions) configuration, but rather a configuration close to it with energy $1/N_1$. We will also require that the different open strings are far apart from each other, i.e. if we denote the average position of these long strings by $X_1^i \cdots X_p^i$ then our analysis will be to first order in $g_{ym}^2 Q^2/(X^i_{p_1} - X^i_{p_j})$. Since we are primarily interested in the scaling with N_1 we will also set for now the size of the circle of the $1+1$ field theory to 2π.

The zeroth order configuration is a configuration of the form

$$X_0^\mu(\sigma) = \begin{pmatrix} X_1^i I_{m_1\times m_1} & & & & \\ & X_2^i\, I_{m_2\times m_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & X_p^i\, I_{m_p\times m_p} \end{pmatrix}, \tag{3.3}$$

where m_i are kept to be a fixed fraction of N_1 as we take $N_1 \to \infty$. We would now like to turn on the hypermultiplets. Since we are dealing with a single brane we will turn on only a single fundamental hypermultiplet, which we will take to live at $\sigma = \pi$. Using the remaining $\Pi\, \mathrm{U}(m_i)$ symmetries we can rotate Q such that only its first component in each block is turned on, and we will also take $\tilde{Q}$ to be of the same form (this will give us a rich enough family of approximate solutions).

Once we have turned on Q, we can choose the matrices at both sides of $\sigma = \pi$ to be

$$X^i\,(\pi-) = X_0^i\,, \qquad X^i\,(\pi+) = X_0^i + g_{ym}\, Q\, \sigma^\mu\, Q^\dagger\,. \tag{3.4}$$

To lowest order in Q and $\tilde{Q}$, most of the eigenvalues of $X^i(\pi+)$ are the same as those of $X^i(\pi-)$ except that one eigenvalue in each block is shifted by order Q^2. It is also clear that we can discuss, to this order, each block separately. Let us focus on the first block. We would like to construct long strings, i.e. complete the configuration to $X^i(\sigma)$ for all σ such that the energy is proportional to $1/m_1$. It is clear that the appropriate long string configuration is such that it begins in, say, $X_1(\pi+)$, winds m_1 times around the circle and ends at $X_1(\pi-)$. The minimal gradients are of order $g_{ym}\, QQ^\dagger/m_1$, and since the length of the long string is m_1,the total energy scales like $1/m_1$ which is the correct scaling to be interpreted as physical state in Matrix theory. This state would be that of a string starting and ending on the same brane.

We can now also discuss the situation in which some impurities are activated and some are not. In the $N_1 = 1$ case this was somewhat problematic because we were looking for these states as wave functions on an exact moduli space. In the large-N_1 limit we relaxed this and it is easy to understand where the additional required states come from. In this case we are allowed to set $Q \neq 0$ at some impurities and $Q = 0$ at the other if we set $Y = 0$ at the first group and $Y \neq 0$ at the second. We can do so while keeping the gradients of Y small (i.e. scaling like $1/N_1$), such that they are compatible with the requirement that the total energy scales as $1/N_1$.

Since we have worked in the Matrix model of the full string theory then this sector contains both the gauge bosons, as well as all the excited string states. However, it is clear that if we mimic the space-time decoupling limit in our model, then we will end up only with the gauge bosons.

One more complication is the following. As explained in [19] the classical moduli space of solution to the D-term equations is actually $4N_1N_4$-dimensional, whereas we have identified a space which can have at most $3N_1N_4$ parameters which corresponds to all the segments of the strings splitting (although for Matrix theory applications we were interested in yet a smaller space). The remaining $N_1 N_4$ parameters are associated with the component A_1 of the gauge field. This is to be expected based on what was explained before that A_1 should actually be viewed as part of a hyper-multiplet. A convenient gauge invariant parameterization of these coordinates can be given by a subset of the quantities

$$Q_i \, \exp^{\int A_1 d\sigma} X^{i_1} \, \exp^{\int A_1 d\sigma} X^{i_2} \cdots Q_{i+1}^{\dagger} \,, \tag{3.5}$$

where the i and $i+1$ index means that we compute the Wilson line between two-neighboring impurities. However for fixed X^i this coordinate is compact, and its kinetic term is multiplied by $1/g_{ym}^2$. Hence the non-homogeneous wave functions along this direction will have an energy proportional to g_{ym}^2 which in spacetime means a mass proportional to $1/g_s$, i.e. they are not perturbative string states. This is familiar from the study of the DLCQ closed string field theory where exciting the Wilson loop in the 16-supercharges $1+1$ SYM corresponds [22] to D-objects.

We have briefly noted before how the Dirichlet boundary conditions come about (i.e. $Q \neq 0$ requires $Y = 0$ on or close to the moduli space). The Neumann boundary conditions come about in the following way. Let us focus on the case of $N_1 = 1$ (which is anyhow similar to what we obtain after we go to the long strings). Using the 2nd line of (3.2) we can now solve for Q in terms of X and insert it into the lagrangian (more precisely, we can solve for Q up to a phase which is a gauge degree of freedom). The resulting lagrangian (placing the "active" impurity at $\sigma = 0$) is

$$\begin{aligned} &\int_{R_-} \left(\partial X^i\right)^2 dt \, d\sigma + \int_{R_+} \left(\partial X^i\right)^2 dt \, d\sigma + \\ &\quad + \frac{1}{g_{ym}} \int f_{ij}(X) \Big(\partial_t X^i(0+,t) - \partial_t X^i(0-,t)\Big) \Big(\partial_t X^j(0+,t) - \partial_t X^j(0-,t)\Big) dt \,, \end{aligned} \tag{3.6}$$

where f is some function which satisfies

$$f(\lambda X) = \frac{1}{\lambda} f(X) \,.$$

The variation of the action with respect to the "boundary terms" gives the equation $\delta X(0+)$ and $\delta X(0-)$ is

$$\delta X(0+)(\partial_\sigma X)(0+) - \delta X(0-)(\partial_\sigma X)(0-) + \frac{1}{g_{ym}}(\cdots) = 0 \,.$$

The details of the last term are irrelevant except that it is non singular for $X \neq 0$, i.e. at generic points along the flat directions. Hence in the limit $g_{ym} \to \infty$ it disappears. It is clear that the path integral contains all values of $X(0\pm)$ and that the different values of $X(0\pm)$ can be connected by physical processes (because this is the case on the moduli space, as measured by its metric). This implies that we

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cannot consistently set $\delta X(0\pm) = 0$. Hence the effective boundary condition that we obtain when the string splits along the impurity is

$$\partial_\sigma X^i\,(0\pm) = 0\,, \tag{3.7}$$

which is the correct Neumann boundary condition.

We have therefore shown how the states in the open string theory, and in particular those of the gauge multiplets in the D4-brane, come about in the Matrix model. And we have studied how their relative position is associated to the value of the impurity variables. Next, we would like to outline how string and SYM perturbation theories, in weak 't Hooft coupling, come about (and what happens to them at large 't Hooft coupling).

3.2 Weak 't Hooft coupling perturbation theory

String and SYM perturbation theories come about now in the following way. The theory contains a large number of different branches, in which different modes of the impurities are excited. Open + closed string perturbation theory comes about as one passes from one branch to another. As we saw above, an excitation of an impurity (i.e. going to the branch $Q \neq 0$) corresponds to opening a hole in the world-sheet where Dirichlet boundary conditions are imposed on coordinates transverse to the brane, and Neumann boundary conditions on those parallel to the brane. If we are starting with some state, then standard quantum mechanical perturbation theory tells us that we have to sum over all ways of exciting an impurity, letting it decay, exciting some other impurity etc. If we start with a closed string state, this corresponds to summing over all ways of cutting holes in the surface. In the finite N_1 system we are allowed to start these holes on a discrete set of points on the surface. However, in the long string picture there are N_4 such points within each interval of size $1/N_1$ of the long string world-sheet. Hence in the large-N_1 limit we can open a hole everywhere in the world-sheet, and summing over all such opening (which is the same as summing over the different ways of exciting the impurities) corresponds to summing over all the moduli of the holes on the world-sheet.

Hence, the open string perturbation theory, i.e. the expansion in the number of holes on the world-sheet is an expansion in the total number of excited impurities. The transition from a branch in which an impurity is not excited to a branch in which it is excited is mediated by some operator, which should give the 't Hooft coupling $g_s\, N_4$ dependence of this transition (similar to [21]). We have not carried out this computation but the factor of N_4 is easy to understand. In the resolved model it merely counts the number of different impurities which can be excited, in an exact correspondence to which brane is the end point of the string. The factor of g_s is associated with exciting a single fixed impurity and therefore there is no additional N_4 dependence.

Jumping ahead, let us now consider what happens when the 't Hooft coupling becomes large. In this case the impurities are excited frequently and an expansion in the number of impurities excited is no longer useful. However, one should now think about the impurities as new "closed string" degrees of freedom in the sense that as $N_1 \to \infty$ the impurities become dense (and evenly spread) on the world-sheet of the long string. These are unusual "closed string" degrees of freedom because, for example, there are no $\partial_\sigma Q$ terms in the lagrangian. Going to the appropriate collective coordinates, we will see that their effective dynamics is that of a closed string moving on the near-horizon limit of the D4-brane.

4. The large 't Hooft coupling limit

We would like to show how the theory of closed strings on the near-horizon limit of the D4-brane comes about from the Matrix model described above, and how the dynamics of the impurity system becomes the dynamics of closed strings. For this purpose, unlike the analysis before, it is useful to first decouple gravity, and only then put the remaining quantum mechanical system into the form of closed strings. The derivation of the decoupled model is carried out in section 4.2 and the derivation of the near-horizon closed string description is carried out in section 4.3. But first we would like to briefly review the near-horizon limit of the D4-branes.

4.1 The near-horizon background

In this subsection we will briefly review this near-horizon background of a cluster of N_4 D4-branes, following [23].

The near-horizon limit appropriate for the D4-brane is:

$$Y^\mu_{nh} = \frac{r^\mu}{\alpha'} = \text{ fixed}\,, \qquad g^2_{D4} = g_s\sqrt{\alpha'} = \text{ fixed}\,, \quad \alpha' \to 0\,, \tag{4.1}$$

where r^μ are the coordinates transverse to the brane, g_{D4} is the Yang-Mills coupling on the D4-brane, and we neglected numerical factors of order 1 (This limit may also be understood as that of an M5-brane wrapped on a circle of size g^2_{D4} which is held fixed as $M_{p,11} \to \infty$ which gives the 6 D(2,0) CFT on a circle).

The corresponding type-IIA background is (in string metric):

$$\begin{aligned} ds^2 &= \alpha' \left(\frac{Y^{3/2}_{nh}}{g_{D4}\sqrt{N_4}}\, dx^2 + \frac{g_{D4}\sqrt{N_4}}{Y^{3/2}_{nh}}\, dY^2_{nh} + g_{D4}\sqrt{N_4 Y_{nh}}\, d\Omega^2 \right), \\ e^\phi &= \left(\frac{Y^3_{nh}\, g^6_{D4}}{N_4} \right)^{1/4} . \end{aligned} \tag{4.2}$$

The type-IIA solution can be trusted in the regime

$$N_4^{-1} \ll g_{D4}^2\, Y_{nh} \ll N_4^{1/3}\,. \tag{4.3}$$

For values of Y_{nh} larger then the upper bound above, the coupling is large and one needs to lift the solution to M-theory, where it asymptotes to the near-horizon limit of the M5-brane [4] compactified on a circle (which reflects the fact mentioned before that the UV fixed point of the D4-brane is actually the $6\,\mathrm{D}(2,0)$ fixed point). For values of Y_{nh} smaller than the lower bound in 4.3 the curvatures become large, i.e. the world-sheet becomes strongly coupled which reflects the fact that the $4+1$ SYM becomes weakly coupled in the IR. It would seem that at the lower end of the region (4.3) the $4+1$ theory is strongly coupled because the $4+1$ effective 't Hooft coupling is large, rather then exhibiting a strong coupling behavior associated with the (2,0) fixed point — the latter takes over at the upper end of this region. Correspondingly the dual description in this regime is given in terms of a string theory.

4.2 The decoupled theory

Equation (2.3) describes the relation between the the parameters of the Matrix model and the parameters of the type-IIA string theory. Hence it is straightforward to follow the decoupling limit (4.1) in the Matrix model. One more useful ingredient is a convenient scaling of the various fields in the $1+1$ impurity lagrangian in this limit. The coordinates X_{-1} and Q_{-1} remain fixed, which implies that we actually need to rescale X (but keep Q fixed). The coordinate Y (or $Y_{-1} = r$) is rescaled according to (4.1) such that Y_{nh} is kept fixed. To summarize we keep fixed the coordinates

$$Q_{-1} = \sqrt{R}\, Q\,, \qquad X_{-1} = \sqrt{R\Sigma}\, X\,, \qquad Y_{nh} = \frac{1}{\sqrt{R\Sigma}\, g_{ym}}\, Y \tag{4.4}$$

while taking the limit

$$\Sigma \longrightarrow 0\,,\ g_{ym} \longrightarrow \infty\,, \quad \Sigma g_{ym}^2 = \frac{R}{g_{D4}^4}\ \text{fixed}\,, \tag{4.5}$$

which, using (2.3) amounts to $M_s \to \infty$ with g_{D4}^2 fixed. In particular we see that Y_{nh} differs from Y by a finite normalization which means that we basically keep Y fixed in this limit. The latter scaling is familiar from other cases in which the near-horizon Coulomb branch is identified with part of the Higgs branch (in a fashion that is set by the hypermultiplet-vector couplings) [8, 9]. Since the size of the circle is also rescaled, it is convenient to choose a new coordinate σ' which remains finite, i.e.

$$\sigma' = \frac{\sigma}{\Sigma}\,. \tag{4.6}$$

Finally we rescale the gauge fields similarly to the coordinates, i.e. A_t is not rescaled, and A_σ is rescaled the same way as ∂_σ.

Using the new quantities the bosonic part of the action becomes (and dropping the (-1) subscript)

$$\begin{aligned}\int dt\, d\sigma' \Bigg(& \frac{1}{R}\left(\mathcal{D}_t X^i\right)^2 + \frac{R}{g_{D4}^4}\left[X^i, Y^\mu_{nh}\right]^2 + R\left(\mathcal{D}_{\sigma'} Y_{nh}\right)^2 + \\ & + D^i\left(\frac{g_{D4}^2}{R}\left(\mathcal{D}_{\sigma'} X\right) + \frac{1}{R}\epsilon_{ijk}\left[X^j, X^k\right]^2\right) + \frac{g_{D4}^2}{R}F^2\Bigg) + \\ & + \frac{\Sigma}{g_{ym}^2}\left(\frac{R^2}{g_{D4}^4}\left(\mathcal{D}_t Y_{nn}\right)^2 + \frac{R^4}{g_{D4}^4}\left[Y_{nh}, Y_{nh}\right]^2 + D^{i^2}\right) + \\ & + \Sigma_k \int dt\, \frac{1}{R}\left(\left(\mathcal{D}_t Q_k\right)^2 + D^i(\sigma_k)\, Q_k\, \sigma^i Q_k^* + \frac{R}{g_{D4}^4} Y_{nh}\,(\sigma_k)^2 Q^2\right), \end{aligned} \tag{4.7}$$

where the sum is over the points of the impurities.

In the limit (4.5) the kinetic terms for the vector multiplet Y drops out, and it should be regarded as an auxiliary variable (in the gauge $A_0 = 0$, F^2 is a kinetic term for a hypermultiplet field). If we integrate it out we obtain the quantum mechanics of the Higgs branch. To obtain the near-horizon geometry, however, we follow the procedure of integrating out the Q hypermultiplets, and describe the model in terms of an effective closed string theory [8, 9].

4.3 The effective action for the vector multiplet

We would like to integrate out the hypermultiplets Q and obtain an effective description for the X and Y fields. Since these are $1+1$ fields, we will obtain a string theory, which of course will be the type-IIA DLCQ Matrix string field theory on the near-horizon geometry of the D4-brane. Since for fixed Y, X and D the Q's appear quadratically, it is straightforward to do the integration. We will expand[4] the result in $\partial Y / Y^2$.

It is convenient to begin with the case that the $1+1$ field theory is a U(1) gauge theory and take the number of D4-branes to infinity. It will then be clear how the $\mathrm{U}(N_1)$ case works even for finite N_4, which is actually our final goal. Again we will be working in the regulated model, in which all the impurities are separated. For the case of the U(1) theory we will also assume that the impurities are scattered more or less uniformly around the circle of the world-sheet.

4.3.1 The U(1) effective action

For a U(1) gauge theory one drops all the commutator terms from the action (4.7). Integrating out the Q variables is straightforward and gives (dropping the decoupled F^2)

[4] At large values of Y, where the string coupling is large, we can still go to the non-abelian form of the Matrix string field theory on the near-horizon limit. We will not, however, be able to go reliably to the long strings picture.

$$\int dt\, d\sigma' \left(\frac{1}{R} (\partial_t X)^2 + R(\partial_{\sigma'} Y_{nh})^2 + \frac{g_{D4}^2}{R} D^i \partial_{\sigma'} X^i \right) +$$
$$+ \Sigma_k \int dt \frac{g_{D4}^2}{R\, Y_{nh}\,(\sigma_k)^3} (\partial_t Y_{nh}(\sigma_k))^2 + \frac{g_{D4}^6}{R^3} D(\sigma_k)^2 \,. \tag{4.8}$$

The next step is to note that as the number of impurities goes to infinity and their location becomes dense on the circle, then we can replace the sum over impurities by an integral. After integrating out the D-term we obtain a string action, whose bosonic part is

$$\int dt\, d\sigma' \left(\frac{1}{R} (\partial_t X)^2 + \frac{R}{g_{D4}^2 N_4} Y_{nh}^3 (\partial_{\sigma'} X)^2 + \right.$$
$$\left. + \frac{g_{D4}^2 N_4}{R} Y_{nh}^{-3} (\partial_t Y_{nh})^2 + R(\partial_{\sigma'} Y_{nh})^2 \right) . \tag{4.9}$$

A string action of this form is somewhat less familiar since it is not in the usual gauge $\gamma^{\alpha\beta} = \delta^{\alpha\beta}$. However, it is easy to read the world-sheet metric and the target space metric from this action. The former is

$$\sqrt{\gamma}\, \gamma^{\alpha\beta} = \begin{pmatrix} \frac{g_{D4} N^{1/2}}{Y^{3/2}} & 0 \\ 0 & \frac{Y^{3/2}}{g_{D4} N^{1/2}} \end{pmatrix} \tag{4.10}$$

(where the 1st component is the time) and the latter is:

$$\frac{Y^{3/2}}{g_{D4}\, N^{1/2}}\, dx^2 + \frac{g_{D4}\, N^{1/2}}{Y^{3/2}}\, dy^2 \tag{4.11}$$

which is the string metric of the D4 background.

It is worth explaining the choice of gauge for the world-sheet metric, Especially since in the gauge (4.10) it is set to a field (Y) dependent value. This is actually natural in the context of light cone quantization in this type of background. Consider the action before fixing reparametrization invariance and the light cone condition. The relevant part for our purposes is

$$\int dt\, d\sigma \sqrt{\gamma}\, \gamma^{\alpha\beta}\, \partial_\alpha X^+ \partial_\beta X^-\, Y^{3/2} \,. \tag{4.12}$$

We would like to impose the gauge $X^+ = \tau$. In order to do that we need that τ will be a solution of the equations of motion for X^+, i.e. $\partial_\alpha Y^{3/2} \sqrt{\gamma}\, \gamma^{\alpha\beta} \partial_\beta \tau = 0$, which implies that we need to set

$$\sqrt{\gamma}\, \gamma^{00} \propto Y^{-3/2} \,.$$

4.3.2 The $\mathrm{U}(N_1)$ case

The Matrix prescription instructs us to take $N_1 \to \infty$, which is what we turn to now. We will then pass to the picture of the long strings, which is again similar to the U(1) case above. This will also clarify why we mandated in the U(1) case (a short string) that the number of impurities goes to infinity and that they are evenly spread around the circle. Whereas this was arbitrary in the U(1) case we will see that this is automatically true for the theory that lives on the long string.

When discussing the U(N_1) theory one needs to restore the commutator terms and the covariant derivatives in (4.7). Furthermore the terms $Y^3(\partial X)^2$ and $Y^{-3}(\partial Y)^2$ will be replaced by non-abelian generalizations. Another change is that we will also generate a term which is roughly of the form

$$\propto \Sigma_k \int dt \, \frac{1}{Y^3} \, [Y, Y]^2 \, .$$

This commutator term in addition to the term $[Y, X]^2$, which is one of the commutators we initially had, tell us that the generic flat directions are when all the X and Y matrices commute. Hence we can go to the "long string" [20, 21], which is a U(1) string with length N_1.

When we go to the long string we are also instructed to concentrate on very long wave length, i.e. on wave lengths of order N_1. Relative to these wave lengths the impurities are dense since the separation between them is of order 1. In the $N_1 \to \infty$ we are justified (even for a finite number of D4-branes) to go to the "continuum impurities" approximation as we did when we went from (4.7) to (4.9), leaving us with a final result (4.9) as the effective dynamics of the long string.

Hence we obtained what we were looking for, i.e. within the impurity model we were able to identify configurations of long strings which are governed by a sigma model on the near-horizon background of the D4-brane.

As in [21, 9] we can also estimate the behavior of the string coupling. To do so we need to identify the mass scale set by the coefficient of the commutator term. The inverse of this mass scale will then determine the string coupling [21]. To correctly identify the mass scale we would like to rescale the coordinates such that the world-sheet metric is (locally) the canonical metric, and then rescale the fields such that their kinetic term is normalized. We will, arbitrarily, focus on the X coordinates. The terms in the lagrangian that contain only X coordinates are (neglecting N_4 and g_{D4} dependence)

$$\int dt \, d\sigma' \Big((\partial_t X)^2 + Y^3 (\partial_{\sigma'} X)^2 + Y^3 [X, X]^2 \Big) \, . \tag{4.13}$$

To go to a Lorentz invariant form with canonically normalized kinetic term we rescale

$$\sigma' = Y^{3/2} \, \sigma'' \, , \qquad X = Y^{-3/4} \, \hat{X}$$

to obtain the action

$$\int dt \, d\sigma'' \, \Big((\partial_t \hat{X})^2 + (\partial_{\sigma''} \hat{X})^2 + Y^{-3/2} [\hat{X}, \hat{X}]^2 \Big) \, . \tag{4.14}$$

We can now read the string coupling from the coefficient of the last term to be

$$g_s(Y) \propto Y^{3/4} \tag{4.15}$$

which is the correct dependence in (4.2).

To conclude, we have used Matrix theory to formulate SYM on D4-branes in a way that contains both open and closed string variables. The closed string variables are auxiliary variables and when we integrate the out, we obtain the open string description and SYM perturbation theory in weak 't Hooft coupling. When we choose to work with a different set of variables, i.e. the closed string variables, we get in a straightforward manner the description of closed strings on the near-horizon geometry (both metric and string coupling). The fact that the two sets of variables are related in a fairly simple way, and that the entire procedure is done in quantum mechanics suggests a simple way of microscopically identifying states in the two descriptions.

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